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Quasi-coherent states for unitary groups

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Abstract. We extend a recent construction of isotopic spin-coherent states to unitary groups. Expectation values in such a quasi-coherent state can be obtained from a generating functional which is given explicitly for the fundamental and the adjoint representations of the unitary groups SU(N) and U(N). The close connection between the corresponding generating functional and the external field problem in QCD is pointed out. Free quantum fields in such a condensed, colour singlet, coherent state are formally related to two-dimensional lattice gauge theories with, in general, a mixed action. In the large-N limit, we exhibit a phase transition for a free quantum field in a singlet quasi-coherent state. A path integral representation of the transition amplitude in terms of quasi-coherent states is also given. The corresponding effective action describes dynamics on a complex Kähler manifold.

1. Introduction

The concept of coherent states plays an important role in various branches of physics (we refer to Klauder and Sudarshan (1968), Glauber (1963, 1964) and Klauder and Skagerstam (1984) for a general introduction to coherent states). Coherent state techniques can for example be used in order to justify the study of classical equations of motion in quantum mechanics or in quantum field theory, i.e. semi-classical methods (Klauder 1977), and in this context we also mention a recent application to Yang–Mills instanton gauge field configuration considerations (Duff and Isham 1980).

Due to their very interesting properties various attempts have been made to generalise the concept of coherent states. One such very successful generalisation to arbitrary Lie groups is due to Perelemov (1972) (see also Gilmore 1972), who constructed what is referred to in the literature as generalised coherent states (see Perelemov (1977) for a review and applications of these states). This construction extends other work on SU(2) spin-coherent states (Radcliffe 1971, Arechi *et al* 1972, Klauder 1963, 1979, 1982, 1983).

Recently an alternative generalisation has been put forward (Skagerstam 1978, 1979, 1980, Eriksson and Skagerstam 1979, 1981, Mukunda *et al* 1981). The corresponding over-complete set of quasi-coherent states transforms according to a given irreducible representation of a compact Lie group under consideration. The construction of these quasi-coherent states is based on the considerations by Bhaumik *et al* (1976), where it was noted that in the case when an Abelian conserved charge is present a complete set of 'coherent states', in which the charge is diagonal, can easily be constructed. In the papers by Skagerstam (1979, 1980) their construction was extended to the field-theoretical situation (for a related discussion see Horn and Silver 1971).

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These U(1)-charged coherent states have recently been applied to a study of the vacuum state in some field-theoretical models by Ni and Wang (1983). An extension of the work by Bhaumik *et al* (1976) to the case of the non-Abelian group SU(2)/Z(2) was given by Skagerstam (1978), Eriksson and Skagerstam (1979, 1981), which recently was also extended to the field-theoretical situation by Mukunda *et al* (1981).

In the present paper we will extend the analysis of Skagerstam (1978), Eriksson and Skagerstam (1979, 1981) and Mukunda *et al* (1981) to higher-dimensional compact Lie groups. We will specialise in the unitary groups $U(N)$ and $SU(N)$, but, as will be clear from our presentation below, the present work can easily be extended to any compact Lie group. We will mainly consider one degree of freedom, as in Skagerstam (1978), Eriksson and Skagerstam (1979, 1981), although most of our results can be extended directly to an arbitrary number of degrees of freedom and will briefly be discussed in the text. Concerning applications of the quasi-coherent states of the form we are considering in the present paper we mention a recent study of the thermodynamics of the non-Abelian, ideal and colourless quark-gluon gas (Skagerstam 1983, 1984). Elsewhere we will return to applications in other areas of physics.

The paper is organised as follows: in § 2 we construct the quasi-coherent states for real representations. We give some specific results for the adjoint representation, which are relevant for the study of condensed gauge fields (Mukunda *et al* 1981). In § 3 we consider complex representations. Some field-theoretical considerations are presented in § 4, where we also include a study of the large-N limit of the colour singlet quasi-coherent states. It is shown that a free quantum field, in a condensed colour singlet state, may exhibit a first-order phase transition if the one-particle state transforms according to the adjoint representation. The origin of this phase transition has a counterpart in the study of some two-dimensional lattice gauge theories (Chen and Zheng 1982, Makeenko and Polikarpov 1982, Samuel 1982, Ogilvie and Horowitz 1983, Jurkiewicz and Zalewski 1983a, b). In § 5 we give some general conclusions of our work including a discussion of the transition amplitude in terms of a quasi-coherent state representation. The effective Lagrangian in the corresponding path integral representation of the transition amplitude describes classical dynamics on a complex Kähler phase space manifold, where the Kähler metric is determined by the generating functional.

2. Quasi-coherent states with real representations

We follow the construction given by Mukunda *et al* (1981) by making use of conventional coherent states. Let D be the dimension of the real representation, $M(g)$, of the group G under consideration and consider D creation (annihilation) operators $a^{\dagger}_{k,\alpha}(a_{k,\alpha})$, $\alpha = 1, \ldots, D$, in Fock space. We introduce in the usual manner the unitary operator

$$U(f) = \exp[(a^{\dagger}, f) - (f^{*}, a)], \tag{2.1}$$

where the D-dimensional vector f describes a one-particle state which transforms according to the $M(g)$ representation and

$$(a^{\dagger}, f) = a^{\dagger}_{k,\alpha} f_{k,\alpha} \tag{2.2}$$

defines a scalar product. The index k can be thought of as a labelling of, for example, the momentum degrees of freedom of the one-particle state. The conventional coherent state $|f\rangle$ is then given by

$$|f\rangle = U(f)|0\rangle, \tag{2.3}$$

where $|0\rangle$ is the Fock-space vacuum and, furthermore,

$$a_{k,\alpha}|f\rangle = f_{k,\alpha}|f\rangle. \tag{2.4}$$

For later purposes we also recall the expression for the overlap between two coherent states (2.3) i.e.

$$\langle f|g\rangle = \exp[(f^*, g) - (f^*, f)/2 - (g^*, g)/2]. \tag{2.5}$$

In the following we will suppress the index k but, when appropriate, it is straightforward to make it explicit.

In order to extract from the coherent state (2.3) the component which transforms according to a given representation $D_{ab}^{\{n\}}$ of the group G, we construct the following quasi-coherent state:

$$|D_{ab}^{\{n\}}; f\rangle = M_{ab}^{\{n\}}(f) \int_G \mathrm{d}g\, D_{ab}^{\{n\}}(g)|M(g)f\rangle \tag{2.6}$$

where, of course, only those representations can occur which can be generated by the representation $M(g)$. In (2.6) dg stands for the invariant Haar-measure on the group (see, for example, Talman 1968) and $M_{ab}^{\{n\}}(f)$ is a normalisation factor which, in general, depends on the representation chosen. $\{n\}$ stands for the sequence of integers, $n_0 \leqslant n_1 \leqslant \ldots \leqslant n_{N-1}$, which characterise the irreducible representation $D_{ab}^{\{n\}}$ (see, for example, Weyl 1949). By its very construction it is now obvious that the state (2.6) transforms irreducibly under group actions which, of course, can be verified by an explicit calculation. The normalisation factor $M_{ab}^{\{n\}}(f)$ can be evaluated by making use of (2.5) i.e.

$$\begin{aligned}\langle f; D_{ab}^{\{n\}}|D_{cd}^{\{n\}}; f\rangle &= |M_{ab}^{\{n\}}(f)|^2 \exp[-(f^*, f)] \int_G \mathrm{d}g\, \mathrm{d}h\, D_{ab}^{\{n\}}(g)^* D_{cd}^{\{n\}}(h) \\ &\quad \times \exp[(f^*, M(g^{-1}h)f)].\end{aligned} \tag{2.7}$$

The invariance of the group measure and the orthogonality relation

$$\int_G \mathrm{d}g\, D_{ab}^{\{n\}}(g) D_{cd}^{\{m\}}(g)^* = \delta(\{n\}, \{m\})\delta_{ac}\delta_{bd}/d_{\{n\}}, \tag{2.8}$$

where $d_{\{n\}}$ is the dimension of the representation $D_{ab}^{\{n\}}$, then leads to the following definition of normalised quasi-coherent states:

$$|D_{ab}^{\{n\}}; f\rangle = \exp[(f^*, f)/2] M_b^{\{n\}}(f)^{-1/2} \int_G \mathrm{d}g\, D_{ab}^{\{n\}}(g)|M(g)\rangle. \tag{2.9}$$

Here $M_b^{\{n\}}(f) \equiv M_{bb}^{\{n\}}(f^*, f)/d_{\{n\}}$, where $M_{ab}^{\{n\}}(h^*, f)$ is given by

$$M_{ab}^{\{n\}}(h^*, f) = \int_G \mathrm{d}g\, D_{ab}^{\{n\}}(g) \exp[(h^*, M(g)f)]. \tag{2.10}$$

$M_b^{\{n\}}(f)$ is a real quantity as expected.

The overlap between two quasi-coherent states (2.9) can be evaluated in an analogous manner with the following result

$$\langle h; D^{\{n\}}_{ab}|D^{\{m\}}_{cd}; f\rangle = \delta(\{m\},\{n\})\delta_{ac}\delta_{bd}M^{\{n\}}_{bb'}(h^*, f)[M^{\{n\}}_{b}(h)M^{\{n\}}_{b}(f)]^{-1/2}. \qquad (2.11)$$

Expectation values of normal ordered operators, invariant under group transformations, can easily be obtained from the generating functional (2.10). Let us illustrate the procedure by considering the number operator

$$N = a^{\dagger}_{k,\alpha}a_{k,\alpha}. \qquad (2.12)$$

Proceeding as in the calculation presented above, we find that

$$\langle f; D^{\{n\}}_{ab}|N|D^{\{n\}}_{ab}; f\rangle = (M^{\{n\}}_{b}(f)d_{\{n\}})^{-1}\int_G \mathrm{d}g\, D^{\{n\}}_{bb'}(g)(f^*, M(g)f)\exp[(f^*, M(g)f)]. \qquad (2.13)$$

The equations (2.10) and (2.13) can then be combined to give

$$\langle f; D^{\{n\}}_{ab}|N|D^{\{n\}}_{ab}; f\rangle = \partial/\partial\lambda \ln[M^{\{n\}}_{bb'}(f^*, \lambda f)]|_{\lambda=1}. \qquad (2.14)$$

The corresponding expression for the dispersion of the number operator, $[\langle\Delta N\rangle_{\{n\},a,b,f}]^2$, has the same formal structure as in the case of quasi-coherent SU(2) states discussed in the literature (Mukunda *et al* 1981) and reads

$$[\langle\Delta N\rangle_{\{n\},a,b,f}]^2 = (\partial^2/\partial\lambda^2 + \partial/\partial\lambda)\ln[M^{\{n\}}_{bb'}(f^*, \lambda f)]|_{\lambda=1}. \qquad (2.15)$$

By making use of the completeness relation

$$\sum_{\{n\},a,b} d_{\{n\}}D^{\{n\}}_{ab}(g)D^{\{n\}}_{ab}(g')^* = \delta(g, g'), \qquad (2.16)$$

we can easily verify (compare the discussion by Skagerstam (1979, 1980), Mukunda *et al* (1982)) the following completeness relation for the quasi-coherent states (2.9)

$$1 = \sum_{\{n\},a,b}\int \mathrm{d}f \exp[-(f^*, f)]M^{\{n\}}_{b}(f)|D^{\{n\}}_{ab}; f\rangle\langle D^{\{n\}}_{ab}; f|. \qquad (2.17)$$

As is the case for the SU(2) quasi-coherent states constructed in Mukunda *et al* (1981), a further representation reduced set of quasi-coherent states can be constructed in a straightforward manner which are labelled only by the Casimir invariants of the group G

$$|\{n\}; f\rangle = \{d_{\{n\}}\exp[(f^*, f)]/M^{\{n\}}(f)\}^{1/2}\int_G \mathrm{d}g\,\chi_{\{n\}}(g)|M(g)f\rangle, \qquad (2.18)$$

where $M^{\{n\}}(f) \equiv M^{\{n\}}(f^*, f)$ and

$$M^{\{n\}}(f^*, h) = \int_G \mathrm{d}g\,\chi_{\{n\}}(g)\exp[(f^*, M(g)h)]. \qquad (2.19)$$

The overlap between two quasi-coherent states (2.18) can easily be computed to be

$$\langle f; \{n\}|\{m\}; h\rangle = \delta(\{n\},\{m\})M^{\{n\}}(f^*, h)/[M^{\{n\}}(f)M^{\{n\}}(h)]^{1/2} \qquad (2.20)$$

and the following completeness relation holds

$$1 = \sum_{\{n\}} d_{\{n\}}\int \mathrm{d}f \exp[-(f^*, f)]M^{\{n\}}(f)|\{n\}; f\rangle\langle f; \{n\}|. \qquad (2.21)$$

Expectation values of G-invariant operators can then be derived in terms of the generating functional (2.19) in a way similar to the one used in the discussion of the quasi-coherent states (2.9).

We realise that in any practical application of the quasi-coherent states (2.9) (some applications have already been indicated by Mukunda *et al* 1981) the analytical structure of the generating functional (2.10) is essential. It turns out however to be a rather difficult problem in itself to evaluate the integral over the group. For reasons of simplicity and because of its practical importance, we now consider the adjoint representation of the group SU(N) (or U(N)) in which case (2.10) takes the following form

$$M(h^*, f) = \int_G \exp[\mathrm{Tr}(h^\dagger g f g^\dagger)/2], \tag{2.22}$$

where

$$f = f_\alpha \lambda_\alpha, \qquad h = h_\beta \lambda_\beta, \tag{2.23}$$

and where λ_α, normalised in such a way that $\mathrm{Tr}(\lambda_\alpha \lambda_\beta) = 2\delta_{\alpha\beta}$, generates the fundamental representation of the SU(N) groups (or U(N)). Integrals of the form (2.22) have actually been considered in the literature in the context of the planar approximation (Itzykson and Zuber 1980) and in the study of chiral U(N)$\otimes$U(N) models (Bars *et al* 1983, Brihaye and Rossi 1983). In terms of a character expansion, (2.22) takes the form ($n_0 = 0$ for $G = \mathrm{SU}(N)$)

$$M(h^*, f) = \sum_{\{n\}} \sigma_{\{n\}}/|n!| d_{\{n\}} \chi_{\{n\}}(h^\dagger) \chi_{\{n\}}(f), \tag{2.24}$$

where $\sigma_{\{n\}}$ is the number of times the representation $D^{\{n\}}(U)$ occurs in the tensor product $\otimes^{|n|} U$ and $|n| = \Sigma_0^{N-1} n_i$. If we consider diagonal elements only, (2.24) can be written in a more explicit form. In this case the matrix f can be diagonalised i.e. it is sufficient to consider f in a diagonal form $f = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$. (2.22) can then be evaluated explicitly with the result (Itzykson and Zuber 1980)

$$M(f^*, f) = \left(\prod_{p=0}^{N-1} p!\right) \det[\exp(\lambda_i \lambda_j^*)] \Big/ \left|\prod_{i>j} (\lambda_i - \lambda_j)\right|^2. \tag{2.25}$$

For the group SU(2), (2.25) reduces to

$$M(f^*, f) = \exp(\lambda_i^* \lambda_i)[1 - \exp(-|\lambda_1 - \lambda_2|^2)]/|\lambda_1 - \lambda_2|^2. \tag{2.26}$$

An analogous expression can be similarly derived for the group SU(3) which for two eigenvalues equal ($\lambda_2 = \lambda_3$) reduces to

$$M(f^*, f) = 2\exp(\lambda_i^* \lambda_i)[1 - |\lambda_1 - \lambda_2|^2 \exp(-|\lambda_1 - \lambda_2|^2) - \exp(-|\lambda_1 - \lambda_2|^2)]/|\lambda_1 - \lambda_2|^4. \tag{2.27}$$

Similar expressions can be derived for higher-dimensional unitary groups.

3. Quasi-coherent states for complex representations

In the present section we study the construction of quasi-coherent states for one-particle states which transform according to a complex representation, $C(g)$, of dimension d. Such a situation occurs when one is considering particles and anti-particles which transform differently under group actions. In practical applications complex representations may for example be associated with fermions (see e.g. Skagerstam 1983, 1984).

In order to take the Dirac–Fermi statistics properly into account one can make use of a coherent state representation of fermionic operators (Klauder 1977, Ohnuki 1977) and a corresponding integration over Grassmannian variables (Berezin 1966). Here we will restrict ourselves to bosonic variables but, as will be clear from the presentation below, it is straightforward to extend our discussion to fermionic variables.

When we consider a complex representation $C(g)$ we must explicitly distinguish between particles and anti-particles. We introduce creation and annihilation operators for particles (anti-particles) $a^{\dagger}_{k,\alpha}(b^{\dagger}_{k,\alpha})$ and $a_{k,\alpha}(b_{k,\alpha})$ where $\alpha = 1, \ldots, d$. Proceeding as in the construction of U(1)-charged quasi-coherent states (Skagerstam 1979, 1980) we then consider the coherent state

$$|f, h\rangle = U_1(f)U_2(h)|0\rangle, \tag{3.1}$$

where U_1 (U_2) corresponds to the U-operator defined by the equation (2.1) for particles (anti-particles). A quasi-coherent state which transforms according to the representation $D^{\{n\}}_{ab}$ is then defined by

$$|D^{\{n\}}_{ab}, f, h\rangle = \exp[(f^*, f)/2 + (h^*, h)/2]C^{\{n\}}_b(f, h)^{-1/2} \int_G \mathrm{d}g\, D^{\{n\}}_{ab}(g)|C(g)f, C(g)^*h\rangle, \tag{3.2}$$

where the normalisation constant $C^{\{n\}}_b(f, h)$ can be evaluated by making use of the procedure given in § 2 with the result $C^{\{n\}}_b(f, h) \equiv C^{\{n\}}_{bb}(h^*, f^*|f, h)/d_{\{n\}}$. Here we have introduced the generating functional

$$C^{\{n\}}_{ab}(h^*_1, f^*_1|f_2, h_2) = \int_G \mathrm{d}g\, D^{\{n\}}_{ab}(g) \exp[\mathrm{Tr}(CF^{\dagger}_{12} + C^{\dagger}H_{12})]. \tag{3.3}$$

In (3.3) the matrix F_{12} has the matrix elements $[F_{12}]_{\alpha\beta} = f^*_{1\alpha}f_{2\beta}$ and similarly for the matrix H_{12}. A completeness relationship can, furthermore, be derived for the quasi-coherent states (3.2) with the following result

$$1 = \sum_{\{n\}} \int \mathrm{d}f\, \mathrm{d}h \exp[-(f^*, f) - (h^*, h)]C^{\{n\}}_b(f, h)|D^{\{n\}}_{ab}; f, h\rangle\langle f, h; D^{\{n\}}_{ab}|. \tag{3.4}$$

Reduction of coherent states in terms of characters can also be carried out in complete analogy with the construction of the quasi-coherent states in terms of characters as discussed in § 2 (cf (2.18)).

Expectation values of operators in the quasi-coherent state (3.2) can be obtained from the generating functional (3.3) in a fashion similar to the discussion of quasi-coherent states with real representations. For the number operator

$$N = a^{\dagger}_{k,\alpha}a_{k,\alpha} + b^{\dagger}_{k,\alpha}b_{k,\alpha}, \tag{3.5}$$

the expectation value becomes

$$\langle f, h; D^{\{n\}}_{ab}|N|D^{\{n\}}_{ab}; f, h\rangle = \partial/\partial\lambda \ln[C^{\{n\}}_{bb}(h^*, f^*|\lambda f, \lambda h)]|_{\lambda=1}. \tag{3.6}$$

For the dispersion of the number operator, $[\langle\Delta N\rangle_{\{n\},a,b,f,h}]^2$, we obtain in analogy with the equation (2.15) the following result

$$[\langle \Delta N\rangle_{\{n\},a,b,f,h}]^2 = [\partial^2/\partial\lambda^2 + \partial/\partial\lambda]\ln[C_{ab}^{\{n\}}(h^*, f^*|\lambda f, \lambda h)], \tag{3.7}$$

evaluated for $\lambda = 1$.

In the case of the fundamental representation of the groups SU(N) and U(N), and for $F \equiv F_{12} = H_{12}$, the generating functional (3.3) has been studied in detail for the singlet representation in the literature either in the context of the external field problem in QCD (Brezin and Gross 1980) or in the context of one-link integrals in the lattice regularisation of QCD (Brower and Nauenberg 1981, Bars 1981, Brower *et al* 1981, Eriksson *et al* 1981, Fateev and Onofri 1981). For the U(N) group the following expression has been derived (Brower *et al* 1981) in terms of the eigenvalues x_i of the matrix FF^+ for the singlet generating functional $C(FF^+) \equiv C(f^*, f^*|f, f)$

$$C(FF^+) = \left(2^{N(N-1)/2}\prod_{p=0}^{N-1} p!\right)\det[\lambda_j^{i-1}I_{i-1}(\lambda_j)]/\det[(\lambda_j^2)^{i-1}], \tag{3.8}$$

where $\lambda_i = 2\sqrt{x_i}$. For the groups SU(N) and U(N) with $N \leqslant 3$ there exists explicit expressions for $C(FF^+)$ in terms of the invariants of the matrix FF^+ (Eriksson *et al* 1981).

4. Field-theoretical considerations

In order to be explicit, we consider free quantum fields which transform according to the fundamental or the adjoint representation. We furthermore restrict ourselves to the singlet representation quasi-coherent states. Remarkably enough, a rather rich structure is exhibited in these rather trivial examples as will be clear from the presentation below.

Let us first consider the fundamental representation. The free field Hamiltonian then reads

$$H_0 = \int d\mu(k)\omega(k)[a_\alpha^+(k)a_\alpha(k) + b_\alpha^+(k)b_\alpha(k)], \tag{4.1}$$

where $d\mu(k) \equiv d^3k/2\omega(k)$ is the Lorentz-invariant phase-space measure. The canonical commutation relations have been normalised in such a way that

$$[a_\alpha(k), a_\beta^+(k')] = 2\omega(k)\delta^3(k-k')\delta_{\alpha\beta}, \tag{4.2}$$

and similarly for the anti-particles. $\omega(k)$ is, of course, the energy of the one-particle state. The scalar product as defined by (2.2) is in terms of the invariant normalisation

$$(a^+, f) = \int d\mu(k)a_\alpha^+(k)f_\alpha(k) \tag{4.3}$$

and the Fock-space states are obtained by the successive actions of the operators (a^+, f) and (b^+, h) on the vacuum state. Let us now consider the singlet quasi-coherent state $|D_{ab}^{\{n\}} = 1; f, f\rangle \equiv |f\rangle_s$ and the expectation value of the free field Hamiltonian H_0 defined above divided by the number of internal degrees of freedom (N^2) i.e.

$$\langle H_0\rangle \equiv {}_s\langle f|H_0|f\rangle_s/N^2 = E_0/2N^2[\lambda^{-1}\partial/\partial\lambda\ \ln Z_\lambda(FF^+)]_{\lambda=1}, \tag{4.4}$$

where $Z_\lambda(FF^\dagger)$ is the (one-link) integral

$$Z_\lambda(FF^\dagger) = \int_G \mathrm{d}g \exp[\lambda \operatorname{Tr}(F^\dagger g + Fg^\dagger)]. \tag{4.5}$$

Here we have for reasons of simplicity assumed that the one-particle overlap integral matrix F in (4.4) satisfies

$$[F]_{\alpha\beta} = \int \mathrm{d}\mu\,(k) f_\alpha^*(k) f_\beta(k) = 2\lambda/E_0 \int \mathrm{d}\mu\,(k)\omega(k) f_\alpha^*(k) f_\beta(k). \tag{4.6}$$

If the matrix F is diagonal, i.e. $[F]_{\alpha\beta} = \lambda N \delta_{\alpha\beta}$, then $Z_\lambda(FF^\dagger)$ takes the following form

$$Z(\lambda) \equiv Z_\lambda(FF^\dagger) = \int_G \mathrm{d}g \exp\{\lambda N[\chi_F(g) + \chi_F(g)^*]\}, \tag{4.7}$$

and (4.3) should be evaluated for an arbitrary λ. E_0 is, for a diagonal F, the expectation value of H_0 in the conventional coherent states $|f, f\rangle$ divided by the number of internal degrees of freedom (N^2). Similarly λ is related to the mean value of the number operator i.e. $2\lambda = \langle f, f|N|f, f\rangle / N^2$. χ_F is the character of the fundamental representation. We recognise in (4.7) the one-plaquette partition function for the two-dimensional lattice gauge theory with Wilson's action and with the gauge group G (Gross and Witten 1980. See also Wadia 1979). In the large-N limit, $Z(\lambda)$ has been computed exactly by means of steepest descent methods with the following results for $\langle H_0 \rangle$

$$\langle H_0 \rangle = \begin{cases} E_0\lambda, & \lambda \leq 0.5 \\ E_0(1 - \frac{1}{4}\lambda), & \lambda \geq 0.5. \end{cases} \tag{4.8}$$

The 'free energy', $-\ln Z(\lambda)$, and its first and second derivatives with respect to λ are continuous at $\lambda = 0.5$. Equation (4.8) exhibits a third-order phase transition since the third derivative of the free energy is discontinuous at $\lambda = 0.5$. The existence of this third-order phase transition actually persists in the case of a general one-particle matrix $[F]_{\alpha\beta}$ (Brezin and Gross 1980). $\langle H_0 \rangle$ as a function of λ is exhibited in figure 1. For finite λ the singlet quasi-coherent state is less condensed as compared to a conventional coherent state $|f, f\rangle$ (cf the discussion by Mukunda *et al* 1981).

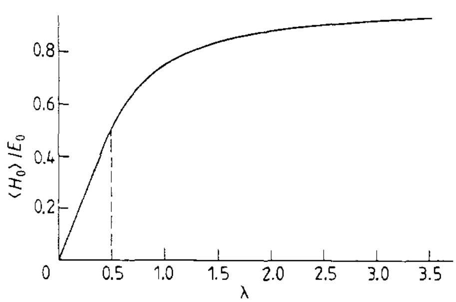

Figure 1. The large-N limit of the expectation value of the free field Hamiltonian, with fields transforming according to the fundamental representation, in a singlet quasi-coherent state. E_0 is defined in the text. $\lambda = 0.5$ corresponds to a third-order phase transition.

The discussion above can easily be carried over to the adjoint representation in which case the free field Hamiltonian reads

$$H_0 = \int d\mu\,(k)\omega(k)a_\alpha^\dagger(k)a_\alpha(k). \tag{4.9}$$

The corresponding expectation value in a singlet quasi-coherent state then becomes

$$\langle H_0\rangle = E_0/N^2[\lambda^{-1}\partial/\partial\lambda \ln Z_\lambda(FF^\dagger)]_{\lambda=1} \tag{4.10}$$

where $Z_\lambda(FF^\dagger)$ is now given by

$$Z_\lambda(FF^\dagger) = \int_G dg \exp\{\tfrac{1}{2}[F]_{\alpha\beta} \operatorname{Tr}(g\lambda_\alpha g^\dagger\lambda_\beta)\}, \tag{4.11}$$

and where we have assumed a relation similar to (4.6). If the matrix F is diagonal, i.e. $[F]_{\alpha\beta} = \lambda\delta_{\alpha\beta}$, equation (4.10) should be evaluated for an arbitrary λ. As can easily be verified, E_0 is then the expectation value of H_0 in the coherent state $|f\rangle$ divided by the number of internal degrees of freedom (N^2) and $\lambda = \langle f|N|f\rangle/N^2$. $Z_\lambda(FF^\dagger)$ then becomes the one-plaquette partition function for two-dimensional lattice gauge theory with Wilson's action in the adjoint representation (Chen and Zheng 1982, Makeenko and Polikarpov 1982, Samuel 1982, Ogilvie and Horowitz 1983, Jurkiewicz and Zalewski 1983a, b) i.e.

$$Z(\lambda) \equiv Z_\lambda(FF^\dagger) = \int_G dg \exp(\lambda\chi_A), \tag{4.12}$$

where χ_A is the character of the adjoint representation. In the large-N limit, $Z(\lambda)$ can be computed exactly with the following result

$$\langle H_0\rangle = \begin{cases} 0, & \lambda < 1 \\ E_0/4[1+(1-1/\lambda)^{1/2}]^2, & \lambda \geq 1. \end{cases} \tag{4.13}$$

In figure 2 we exhibit $\langle H_0\rangle$ as a function of λ. In the large-N limit we therefore obtain a first-order phase transition for one-particle states transforming according to the adjoint representation. For $\lambda < 1$ there are not sufficiently many states available to form a non-trivial singlet state and, hence, the only attainable state is the vacuum state.

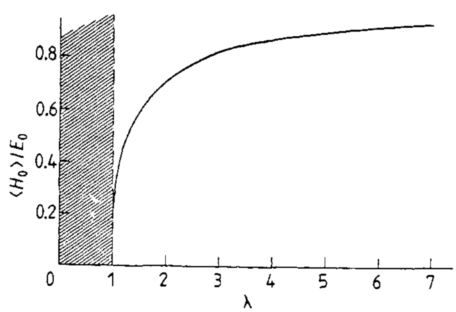

Figure 2. The large-N limit of the expectation value of the free field Hamiltonian, where the fields transform according to the adjoint representation, in a singlet quasi-coherent state. E_0 is defined in the text. For $\lambda < 1$, the shaded area in the figure, the only attainable state is the zero-energy state due to the existence of a first-order phase transition at $\lambda = 1$.

One can trace the presence of the first-order phase transition back to the spontaneous breaking of the global symmetry under the action of the central elements of SU(N). In the large-N limit the central elements, Z_N, form a U(1) sub-group which, for $\lambda \geq 1$, is spontaneously broken. This breaking of the central U(1) symmetry is similar to the breaking cf the $\mathrm{U}(1)^d$-symmetry breaking in the quenched Eguchi–Kawai model (Eguchi and Kawai 1982, Bhanot *et al* 1982). It is amazing to notice that the existence of a first-order phase transition for an ideal, colourless gluon gas in the large-N limit (Skagerstam 1983, 1984) has a formal counterpart in the first-order phase transition exhibited in (4.13).

More generally we can, of course, consider a free field Hamiltonian with, for example, one set of fields transforming according to the fundamental representation and another set of fields which transforms according to the adjoint representation. The generating functionals (4.7) and (4.11) will then combine to yield the one-plaquette partition function for the two-dimensional lattice gauge theory with the mixed action (Chen and Zheng 1982, Makeenko and Polikarpov 1982, Samuel 1982, Ogilvie and Horowitz 1983, Jurkiewicz and Zalewski 1983a, b). Clearly, the analysis can be extended to any representation. The large-N limit of the corresponding generating functional has been studied to some extent in the literature (Yee 1983). The general structure which emerges from such a study is a phase-diagram with, in general, first and higher-order phase transitions in the sense defined above.

For interacting fields a perturbative scheme can, of course, be developed in terms of quasi-coherent states. We also mention the possibility of performing a variational calculation for interacting fields along the lines suggested by Ni and Wang (1983).

5. Final remarks

Coherent states have the property that they closely describe the classical dynamics of a given dynamical system under consideration (Klauder and Sudarshan 1968, Glauber 1963, 1964). In the present paper we have considered quasi-coherent states appropriate for dynamical systems where an additional constraint on the dynamics is present, namely the existence of a conserved and, in general, non-Abelian charge.

It has been pointed out by Klauder (1978, 1979, 1982, 1983) and Kuratsuji and Suzuki (1980a, b, 1981, 1983) (see also in this context Shankar 1980) that a path integral representation of the transition amplitude in terms of generalised coherent states may be useful in order to study the relationship between the quantum and classical aspects of a dynamical system with a 'curved phase space' as in the case of a spin system. In such a context it is natural to extend these considerations by making use of quasi-coherent states. Let H be a Hamiltonian which commutes with the generators of a Lie group G and let $|\psi\rangle$ denote a corresponding quasi-coherent state with the following decomposition of the unity operator

$$1 = \int \mathrm{d}\psi |\psi\rangle\langle\psi|, \tag{5.1}$$

where we have suppressed all group-theoretical indices. A path integral representation of the transition amplitude

$$T_{12} \equiv T(\psi_2, t_2; \psi_1, t_1) = \langle\psi_2| \exp[-\mathrm{i}/\hbar H(t_2 - t_1)]|\psi_1\rangle \tag{5.2}$$

can then easily be written down

$$T_{12}=\lim_{n\to\infty}\int\ldots\int\prod_{k=1}^{n-1}\mathrm{d}\psi_k\prod_{l=1}^{n}\langle\psi_l|\psi_{l-1}\rangle\prod_{m=1}^{n}[1-\mathrm{i}\varepsilon/\hbar\langle\psi_m|H|\psi_{m-1}\rangle/\langle\psi_m|\psi_{m-1}\rangle] \tag{5.3}$$

where $\varepsilon=(t_2-t_1)/n$. We now specifically consider a real representation, $M(g)$, and quasi-coherent states of the form (2.18) i.e. states which are diagonal in terms of the Casimir invariants. Other cases can be treated similarly. By making use of (2.19) and (2.20) we derive

$$\langle f+\Delta f;\{n\}|\{n\};f\rangle=\exp\{\tfrac{1}{2}[(\Delta f^*,g^{\{n\}}f)-(f^*,g^{\{n\}}\Delta f)]\}[1+\mathrm{O}((\Delta f)^2)] \tag{5.4}$$

where we have defined a metric

$$g_{\alpha\beta}^{\{n\}}(f^*,f)=M^{\{n\}}(f^*,f)^{-1}\int_G \mathrm{d}g\chi_{\{n\}}(g)M_{\alpha\beta}(g)\exp[(f^*,M(g)f)]. \tag{5.5}$$

The transition amplitude T_{12} can then be written as follows

$$T_{12}\equiv\langle f_2;\{n\};t_2|t_1;\{m\};f_1\rangle$$
$$=\delta(\{n\},\{m\})\int D[f(t)]\exp\{\mathrm{i}/\hbar S[f^*(f),\dot f^*(t),f(t),\dot f(t)]\}, \tag{5.6}$$

where the action S is defined by

$$S=\int_1^2 \mathrm{d}t\,\mathscr{L}[f^*(t),\dot f^*(t),f(t),\dot f(t)]. \tag{5.7}$$

The Lagrangian density $\mathscr{L}$ is defined by the Legendre transformation

$$\mathscr{L}=\mathrm{i}\hbar/2[(f^*,g^{\{n\}}\dot f)-(\dot f^*,g^{\{n\}}f)]-\mathscr{H}(f^*,f) \tag{5.8}$$

and

$$\mathscr{H}(f^*,f)=\delta(\{n\},\{m\})\langle f;\{n\}|H|\{m\};f\rangle. \tag{5.9}$$

The functional integration measure $D[f(t)]$ is derived by making use of the decomposition of the unity operator (2.21) i.e. formally the measure has the form

$$D[f(t)]=\prod_t\sum_{\{n\}}\mathrm{d}f(t)\exp[-(f^*(t),f(t))]M^{\{n\}}(f^*(t),f(t))\,d_{\{n\}}. \tag{5.10}$$

The presence of the metric $g_{\alpha\beta}^{\{n\}}(f^*,f)$ in the 'kinetic' part of $\mathscr{L}$ is related to the geometrical structure of the phase space in a fashion similar to the complex Kähler manifold derived by making use of the SU(2) generalised spin-coherent states (Klauder 1978, 1979, 1982, 1983, Kuratsuji and Suzuki 1980a, b, 1981, 1983). The equations of motion derived from the action (5.7) in general take the following form

$$\mathrm{i}\hbar G_{\alpha\beta}^{\{n\}}(f^*,f)\dot f_\beta^*=\partial\mathscr{H}/\partial f_\alpha,\qquad \mathrm{i}\hbar G_{\alpha\beta}^{\{n\}}(f^*,f)\dot f_\beta=\ -\partial\mathscr{H}/\partial f_\alpha^* \tag{5.11}$$

where

$$G_{\alpha\beta}^{\{n\}}(f^*,f)=\partial^2\ln M^{\{n\}}(f^*,f)/\partial f_\alpha\partial f_\beta^* \tag{5.12}$$

defines a metric on a Kähler manifold (Kobayashi and Nomizu 1969).

Semi-classical aspects on the dynamical system defined by the Langrangian (5.8) have been discussed in the literature (Kuratsiju and Suzuki 1980a, b, 1981, 1983) and we will not dwell further on this question in the present paper. Here we only make the following observations. If H describes the interaction of a Yang–Mills particle with an external gauge field (Balachandran *et al* 1983), a path integral representation of T_{12} in terms of conventional coherent states yields an effective action of the form (5.8) with $g^{\{n\}}_{\alpha\beta} = \delta_{\alpha\beta}$. The corresponding c-number Lagrangian has been studied in detail in the literature (Balachandran *et al* 1977), where it was observed that the corresponding canonical quantisation in general leads to reducible representations of the group G. Allowing for only one irreducible representation to contribute in the derivation of (5.6) therefore corresponds to the non-trivial form of the metric $G^{\{n\}}_{\alpha\beta}$ (or $g^{\{n\}}_{\alpha\beta}$). The canonical quantisation of the dynamical system described by the Lagrangian (5.8) must, of course, yield only one irreducible representation of the group G.

As a final remark we notice that the probability, $P(\Delta E)$, for the emission of soft non-Abelian massless vector bosons up to a certain total energy ΔE from a classical c-number source, neglecting self-interactions among the vector bosons, can be computed along the lines of Skagerstam (1979, 1980). By making use of the asymptotic form of equation (2.25) it can then be verified that one obtains an infrared finite result for $P(\Delta E)$ with a structure similar to the corresponding expression in quantum electrodynamics (cf Mukunda *et al* 1981 and references cited therein).

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